20. Let F be a field of characteristic 2, and let K be an extension of F of degree 2.
   (a) Prove that K has the form $F(\alpha)$, where α is the root of an irreducible polynomial over F of the form $x^2 + x + a$, and that the other root of this equation is $\alpha + 1$.
   (b) Is it true that there is an automorphism of K sending $\alpha \rightsquigarrow \alpha + 1$?

2. *Cubic Equations*

1. Prove that the discriminant of a real cubic is positive if all the roots are real, and negative if not.
2. Determine the Galois groups of the following polynomials.
   (a) $x^3 - 2$ (b) $x^3 + 27x - 4$ (c) $x^3 + x + 1$ (d) $x^3 + 3x + 14$
   (e) $x^3 - 3x^2 + 1$ (f) $x^3 - 21x + 7$ (g) $x^3 + x^2 - 2x - 1$
   (h) $x^3 + x^2 - 2x + 1$
3. Let f be an irreducible cubic polynomial over F, and let δ be the square root of the discriminant of f. Prove that f remains irreducible over the field $F(\delta)$.
4. Let α be a complex root of the polynomial $x^3 + x + 1$ over $\mathbb{Q}$, and let K be a splitting field of this polynomial over $\mathbb{Q}$.
   (a) Is $\sqrt{-3}$ in the field $\mathbb{Q}(\alpha)$? Is it in K?
   (b) Prove that the field $\mathbb{Q}(\alpha)$ has no automorphism except the identity.

*5. Prove Proposition (2.16) directly for a cubic of the form (2.3), by determining the formula which expresses α_2 in terms of α_1, δ, p, q explicitly.

6. Let $f \in \mathbb{Q}[x]$ be an irreducible cubic polynomial which has exactly one real root, and let K be its splitting field over $\mathbb{Q}$. Prove that $[K : \mathbb{Q}] = 6$.
7. When does the polynomial $x^3 + px + q$ have a multiple root?
8. Determine the coefficients p, q which are obtained from the general cubic (2.1) by the substitution (2.2).
9. Prove that the discriminant of the cubic $x^3 + px + q$ is $-4p^3 - 27q^2$.

3. *Symmetric Functions*

1. Derive the expression (3.10) for the discriminant of a cubic by the method of undetermined coefficients.
2. Let $f(u)$ be a symmetric polynomial of degree d in $u_1, \ldots, u_n$, and let $f^0(u_1, \ldots, u_{n-1}) = f(u_1, \ldots, u_{n-1}, 0)$. Say that $f^0(u) = g(s^0)$, where s_i^0 are the elementary symmetric functions in $u_1, \ldots, u_{n-1}$. Prove that if $n > d$, then $f(u) = g(s)$.
3. Compute the discriminant of a quintic polynomial of the form $x^5 + ax + b$.
4. With each of the following polynomials, determine whether or not it is a symmetric function, and if so, write it in terms of the elementary symmetric functions.
   (a) $u_1^2u_2 + u_2^2u_1$ $(n = 2)$
   (b) $u_1^2u_2 + u_2^2u_3 + u_3^2u_1$ $(n = 3)$
   (c) $(u_1 + u_2)(u_2 + u_3)(u_1 + u_3)$ $(n = 3)$
   (d) $u_1^3u_2 + u_2^3u_3 + u_3^3u_1 - u_1u_2^3 - u_2u_3^3 - u_3u_1^3$ $(n = 3)$
   (e) $u_1^3 + u_2^3 + \cdots + u_n^3$
5. Find two natural bases for the ring of symmetric functions, as free module over the ring R.